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Notes on wave theory in heat conduction : a new boundary condition

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Abstract—The physical basis behind the simple hyperbolic heat transport model is discussed. The model equations are interpreted as energy equations for a system in a state of local nonequilibrium. On this basis a new boundary condition for the hyperbolic model is proposed when the temperature of a surface is given. It is based on a nonequilibrium situation and it reduces to the conventional boundary condition under the same assumptions as needed to justify Fourier's law of heat conduction and reduces to the known temperature-jump boundary condition at steady-state conditions. The significance of the new boundary condition is demonstrated through its application to one-dimensional heat transfer problems via hyperbolic equations. The difference between solutions of hyperbolic and Fourier models is found to be much larger than reported before and it does not vanish in the steady-state. © 1997 Elsevier Science Ltd.

INTRODUCTION

The classical theory of heat conduction is based on Fourier's constitutive relation : $q = -k\,\partial T/\partial x$. Fourier's equation with a constant heat conductivity k together with the equation of energy conservation :

$$c\frac{\partial T}{\partial t} + \frac{\partial q}{\partial x} = 0 \qquad (1)$$

gives the conventional parabolic heat conduction equation

$$\frac{\partial T}{\partial t} = \frac{k}{c}\frac{\partial^2 T}{\partial x^2}. \qquad (2)$$

Several boundary conditions for equation (2) are often used. One of the commonly encountered conditions at the boundary surface $x = 0$ with a prescribed temperature T_s is :

$$T(0, t) = T_s. \qquad (3)$$

To circumvent the known deficiencies of Fourier's law in the description of problems involving a high rate of temperature change the concept of heat transmission by waves has been introduced [1–4]. The most simple and well known alternative to the Fourier law of heat conduction for the one-dimensional (1-D) case is given by the equation :

$$q + \tau\frac{\partial q}{\partial t} = -k\frac{\partial T}{\partial x}. \qquad (4)$$

This equation together with equation (1) forms a hyperbolic system of equations, which describes a heat propagation with a finite speed $v = (k/c\tau)^{1/2}$. The significance of equation (4) has been demonstrated for different practical applications [5–8]. Assuming constant properties we may combine equations (1) and (4) to form the often used dissipative wave equation :

$$\tau\frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \frac{k}{c}\frac{\partial^2 T}{\partial x^2}. \qquad (5)$$

Numerous studies in the literature have considered the solutions of a variety of problems via hyperbolic heat conduction equations. Almost all solutions reported were obtained by the use of the same boundary conditions as known for the classical Fourier model.

Boundary conditions express the interaction of the system at hand with the surroundings and can never be universal. They are a constituent part of the transport model and should be consistent with the description of the heat transport inside the system. Parabolic and hyperbolic equations have different properties, they obviously imply entirely different physical situations and the question arises whether the known boundary conditions are physically realistic for an essentially different situation and equation. Moreover, it is known that the boundary condition (3) is not always appropriate. In particular, it becomes inaccurate to describe the heat transfer to a body in a rarefied gas and should be replaced by the temperature-jump condition [9] :

$$T(0, t) - T_s = \frac{2-a}{a}\lambda\frac{\partial T(0, t)}{\partial x} \qquad (6)$$

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NOMENCLATURE

a	accomodation coefficient	δ	dimensionless distance, $L/(2\sqrt{\alpha\tau})$
c	heat capacity per unit volume	ζ	dimensionless time, $t/(2\tau)$
e	internal energy per unit volume	θ	dimensionless temperature, defined by
I_n	modified Bessel function of order n		equations (17) or (23)
k	thermal conductivity	λ	carrier mean free path
L	distance between the surfaces	τ	relaxation time
Q	dimensionless heat flux, defined by	χ	dimensionless position, $x/(2\sqrt{\alpha\tau})$.
	equations (17) or (23)		
q	heat flux		
T	temperature		
t	time	**Subscripts**	
x	position	in	initial
U	dimensionless accumulated heat	L	at $x = L$
v	mean speed of the thermal carriers in	s	surface
	one direction, thermal propagation	0	at $x = 0$
	speed.	1, 2	first and second thermal waves
Greek symbols		*	at equilibrium
α	thermal diffusivity, k/c	∞	at $\lambda/L \to \infty$.

where λ is a distance approximately equal to the molecular mean free path. Bai and Lavine [10] have found a large difference between the solutions of equation (5) with and without the jump boundary condition. One may wonder, what condition is better and why.

In this paper we present an attempt to answer these questions through consideration of the physical background behind the hyperbolic equations. Equations (1) and (4) are interpreted as a linear combination of the energy equations for a system at a state of local nonequilibrium. This leads to a new, physically more meaningful boundary condition for hyperbolic equations when the temperature of the surface surrounding the system is given. This condition can be envisaged as a nonequilibrium condition. At steady-state it reduces to the temperature-jump boundary condition (6). The conventional, equilibrium boundary condition (3) is obtained from the nonequilibrium one under the same assumptions as needed to justify Fourier's law of heat conduction. The significance of the new boundary condition for the hyperbolic model will be illustrated with examples involving steady and unsteady heat conduction in a finite and semi-infinite slab and a comparison to the available solutions of hyperbolic equation (5) and the classical Fourier model. The use of the new boundary condition for hyperbolic equations avoids the transient temperature rise exceeding the initial and boundary temperatures as found by Taitel [11] and temperatures below absolute zero as reported by Bai and Lavine [10, 12]. The difference between the solutions of the hyperbolic and Fourier models is much larger than the difference found in previous publications. The hyperbolic model with the new boundary condition is fundamentally

different from the parabolic one anywhere, also in the steady-state. In particular, for a pure wave equation the new boundary conditions allow us to establish a finite, steady heat flow between parallel plates at different temperatures what is impossible with the boundary conditions of the classic Fourier model.

INTERPRETATION OF THE HYPERBOLIC HEAT CONDUCTION EQUATIONS

Most theoretical developments in heat conduction are intended to generalize the available results. Our aim is different, we wish to obtain a physically realistic and clear picture of the phenomena. We shall simplify the problem as much as possible, maintaining only those features without which heat flow becomes impossible.

We first state that the microscopic origin of the heat flux is the random motion of thermal energy carriers like electrons, atoms, molecules, chaotic convective streams or their combinations. The following simplifications are introduced. First, a 1-D problem is considered, i.e. all quantities are regarded as variable only in one direction, say x. Second, the values of the carrier's velocities are constant and equal to the average speed of carriers in x-direction. Third, the carriers as a whole are at rest: as a consequence half of the carriers moves in the position x-direction and the other half in the negative one. We can group the carriers into two streams: one consists of the carriers moving in the positive x-direction, and the other of those moving in the opposite direction. We will not calculate the parameters of the equations and, therefore, the cause of their movement is not important to us. Let e_1 and e_2 be the internal energy densities of the

carriers moving to the right and to the left per unit volume occupied by these carriers. By internal energy we understand any appropriate kind of energies associated with the carriers. The average internal energy density and the heat flux can be found as:

$$e = (e_1 + e_2)/2 \quad q = v(e_1 - e_2)/2. \tag{7}$$

The energy equations for each group of carrier are:

$$\frac{\partial e_1}{\partial t} + v\frac{\partial e_1}{\partial x} = F_1(e_1, e_2) \tag{8}$$

$$\frac{\partial e_2}{\partial t} - v\frac{\partial e_2}{\partial x} = F_2(e_1, e_2) \tag{9}$$

where F_1 and F_2 describe the variations of the internal energies e_1 and e_2 due to energy exchange between carriers moving in opposite directions. These source terms may depend on e_1 and e_2 and on the nature of the process under consideration. In view of the energy conservation $F_1 = -F_2 = F(e_1, e_2)$. The dependence of F on e_1 and e_2 can be represented as its dependence on the average internal energy $e = (e_2 + e_1)/2$ and the difference of the energy densities $e_1 - e_2$. The simplest dependence of $F(e, e_1 - e_2)$ on $e_1 - e_2$ is given by the linear function:

$$F(e_1, e_2) = \frac{e_2 - e_1}{2\tau} \tag{10}$$

where τ is a parameter having units of time; it may depend on the properties, the state of the system under consideration and on the average internal energy e; the coefficient $1/2$ is introduced for convenience. The constant term in this function, independent on $e_1 - e_2$, is zero since equation (8) and (9) should also be valid for a state of global equilibrium. Equation (10) can be viewed as a Taylor expansion of the energy exchange rate in the neighbourhood of the equilibrium position. Equation (10) to be physically realistic implies that τ is positive.

Equations (8)–(10) describe the energy transport in terms of internal energies of different groups of the carriers. Taking into account equations (7) they can be rewritten in terms of e and q and the following discussion can be given using only these variables. However, for practical purposes it is convenient to use the heat capacity and the temperature of the material instead of the internal energy $e = (e_1 + e_2)/2$. In our consideration of the state of the system at each point the energies of the carriers moving in opposite directions are different. This means that we deal with a state of local nonequilibrium and that we face a disputable problem: what is nonequilibrium temperature? Different versions of nonequilibrium thermodynamics treat this problem in different ways [13, 14]. In our representation of the local nonequilibrium state it is reasonable to introduce the temperatures T_1 and T_2 of each group of carriers exactly in the same way as for the whole system in equilibrium. The temperature of a group of carriers is the temperature of the equilibrium

system if its internal energy were equal to the internal energy of this group, so $e_1 = e_*(T_1)$ and $e_2 = e_*(T_2)$, where $e_*(T)$ is the internal energy density for the system at equilibrium with a temperature of T. It is further assumed that the equilibrium internal energy depends linearly on the temperature, according to:

$$e_*(T) = e_0 + c(T - T_0) \tag{11}$$

where c is the heat capacity, being independent of time and position, and e_0 a constant reference internal energy. After this, equations (8)–(10) take the form:

$$\frac{\partial T_1}{\partial t} + v\frac{\partial T_1}{\partial x} = \frac{T_2 - T_1}{2\tau} \tag{12}$$

$$\frac{\partial T_2}{\partial t} - v\frac{\partial T_2}{\partial x} = -\frac{T_2 - T_1}{2\tau}. \tag{13}$$

By the use of equations (7) and (11) the temperatures of the groups of carriers can be written in terms of the temperature and heat flux as:

$$T_1 = T + \frac{q}{cv} \quad T_2 = T - \frac{q}{cv}. \tag{14}$$

Adding and subtracting equations (12) and (13) and using equation (14) we obtain exactly equations (1) and (4) provided $k = cv^2\tau$. With such an interpretation the velocity of the heat waves $(k/c\tau)^{1/2}$ is equal to the mean carrier speed in one direction and the parameter τ is the characteristic time required to attain equilibrium between the carriers.

The consideration above demonstrates that it is impossible to obtain a heat conduction equation simpler than equations (1) and (4) without physically unrealistic assumptions. If we assume $\tau \to 0$ we do not obtain the classical Fourier law because in this case the heat conductivity $k = cv^2\tau$ also becomes zero. To obtain Fourier's law from equation (4) we should keep the heat conductivity finite and, therefore, assume the carrier's speed is infinite or accept a physical contradiction.

The described approach to heat conduction is flexible and can also be applied with less restrictive simplifying assumptions. One can find interesting modifications of the governing equations, if the equilibrium internal energy density $e_*(T)$ is a nonlinear function of the temperature, if the energy exchange rate $F(e_1, e_2)$ is not a linear function of $e_1 - e_2$ or if volumetric, temperature dependent heat sources are present. If several mechanisms of heat transport with different relaxation times and carrier velocities play a role, governing equations similar to those known for viscoelsatic fluids [15] are obtained, although not the Jeffreys type models. Also, one can find how the model changes with spatial variations of the system properties and when the carrier velocity, relaxation time and/or heat capacity are temperature dependent. In consideration of multidimensional transport qualitatively more complicated problems arise, because of the infinite number of possible directions of movement

of carriers. A vector form of equations (1) and (4) can be justified only for small deviations from equilibrium of the state of the system. For the present discussion we do not need to go to a further generalization of equations (1) and (4).

INITIAL AND BOUNDARY CONDITIONS FOR HYPERBOLIC HEAT CONDUCTION EQUATIONS

Equations (1) and (4) contain two state variables, the temperature and the heat flux, and need two initial and two boundary conditions. Both initial conditions are clear, they are the specification of the temperature and the flux at some moment of time. From the previous considerations it follows $q = cv(T_1 - T_2)/2 = cv(T_0 - T_2)$ and $T_2 > 0$. Therefore, an initial value of the heat flux larger than cvT_0 is physically impossible.

The simplified interpretation of hyperbolic equations given above is helpful for the formulation of the boundary conditions when the temperature of the surrounding surface is known. Equations (1) and (4) imply two waves or two groups of carriers. One wave moves away from the surface and the second one moves towards the surface. Physically only the first wave, moving away from the surface, can have been influenced by the surface. The second wave, moving toward the surface, 'knows' about the surface only by the reflected wave, as described by the right-hand side of equations (8) and (9). Its temperature everywhere is in between the temperature of the reflected wave and the temperature far away from the wall. The larger the relaxation time compared to the characteristic process time, the less the second wave 'knows' about the surface; it 'knows' nothing about the surface in the limit of an infinitely large relaxation time. Similar arguments are used in the kinetic gas theory to estimate the speed of slip and the temperature drop at the wall [16].

If we assume initially that the reflected wave has come into thermal equilibrium with the surface and has a mean energy as corresponds to the surface temperature, we may write the boundary condition for material situated in the region $x > 0$ as:

$$T_1(0, t) = T_s \quad \text{or} \quad T(0, t) + \frac{q(0, t)}{cv} = T_s \quad (15)$$

where T_1 is the temperature of the wave moving away from the surface. In contrast to the conventional, equilibrium boundary condition (3) this boundary condition of equation (15) can be viewed as a nonequilibrium one and it states the temperature of the surface is different from that in the adjacent layer. This difference is determined by the departure from the equilibrium state near the surface and is characterized by the value of the heat flux. In the limit of $\tau \to 0$ and $\alpha = \tau v^2$ is constant, leading to Fourier's law, the boundary condition (15) transforms into the classical boundary condition (3).

From experiments and the kinetic gas theory it is known that the assumption, that molecules leaving the wall have come into thermal equilibrium with it, is usually not justified. It is necessary to introduce a fractional quantity known as the 'accommodation coefficient' to represent the extent to which the colliding molecules adjust their mean energy or temperature to that of the wall [9, 17]. The same situation may be predicted for systems not being gaseous. Introducing an accommodation coefficient a by analogy with that in the kinetic gas theory as the ratio of the actual mean-energy change of carriers interacting with a surface to the mean-energy change, if the carriers had come into equilibrium with the surface, we may write the boundary condition in a more general way than in equation (15):

$$T_1(0, t) - T_2(0, t) = a(T_s - T_2(0, t))$$

$$\text{or} \quad T(0, t) + \frac{2-a}{a} \frac{q(0, t)}{cv} = T_s. \quad (16)$$

At steady-state $q = -k \, \partial T/\partial x$ and equation (16) takes the well known form of the jump boundary condition, equation (6), with $\lambda = v\tau$.

This consideration does not influence the conventional boundary condition where the heat flux crossing the surface is prescribed. Specification of the heat flux at the surface implies the temperature of the reflected wave is such that the difference of the energies transmitted by two waves is equal to the prescribed heat flux.

The prescribed surface temperature is a special, limiting case of two bodies brought into contact. The nonequilibrium boundary conditions (15) and (16) can easily be extended to the more general situation where heat transfer from both sides of the interface should be considered. In this case two boundary conditions are necessary, one condition is obvious. The equality of the fluxes at the interface, although it does not mean that $q(-0) = q(+0)$ everywhere; the heat flux can be discontinuous under dynamic conditions. To formulate the second condition we must express the energies or temperatures of the waves leaving the surface through the energies of the arriving waves. This energy exchange obviously depends on the specific properties of the bodies brought into contact and can be characterized by introducing an accommodation coefficient similar to that in equation (16). Heat generation at the surface, as for the Stefan problem, can also be easily incorporated into the energy balance at the interface. The most simple situation occurs if two identical media are brought in perfect contact. In this case, the situation at the interface is indistinguishable from that in the body of materials and heat waves will cross the interface without any change. To elucidate the significance of the nonequilibrium boundary condition some examples are given below assuming that the accommodation coefficient $a = 1$.

EXAMPLES OF THE APPLICATION OF THE NEW BOUNDARY CONDITION

Steady heat conduction in a finite slab

Let us consider a body conducting heat between two parallel plates at different temperatures T_0 and T_L and separated by a distance L. We shall compare the solutions of equations (1) and (4) or (12) and (13) using different boundary conditions and the solution of classical Fourier's model. Define, for convenience, the following dimensionless variables:

$$\theta = \frac{T - T_L}{T_0 - T_L} \quad Q = \frac{q\sqrt{\alpha\tau}}{k(T_0 - T_L)} \quad \chi = \frac{x}{2\sqrt{\alpha\tau}}. \quad (17)$$

With these quantities, the temperature and the heat flux are determined through the temperatures of the waves as:

$$\theta = (\theta_1 + \theta_2)/2 \quad Q = (\theta_1 - \theta_2)/2.$$

The steady-state solution of the equations (12) and (13) with the conventional boundary conditions:

$$\chi = 0 \quad \theta = 1, \quad \chi = \delta \quad \theta = 0 \quad (18)$$

is

$$\theta_1 = \frac{\delta - \chi + 1/2}{\delta} \quad \theta_2 = \frac{\delta - \chi - 1/2}{\delta}$$

$$\theta = \frac{\delta - \chi}{\delta} \quad Q = \frac{1}{2\delta} \quad (19)$$

where the dimensionless distance $\delta = L/(2\sqrt{\alpha\tau})$ has the same meaning as the inverse Knudsen number for gases. The temperature distribution and the heat flux in this case are the same as predicted with the Fourier law.

When the new boundary conditions:

$$\chi = 0 \quad \theta_1 = 1 \quad \chi = \delta \quad \theta_2 = 0 \quad (20)$$

are used, the corresponding solution is

$$\theta_1 = \frac{1 + \delta - \chi}{1 + \delta} \quad \theta_2 = \frac{\delta - \chi}{1 + \delta}$$

$$\theta = \frac{1/2 + \delta - \chi}{1 + \delta} \quad Q = \frac{1}{2(1 + \delta)}. \quad (21)$$

This example shows that with the new boundary conditions the predictions of the parabolic and hyperbolic models also differ under steady-state conditions. The difference is insignificant when $\delta \gg 1$ and it grows as δ decreases. The new solution predicts the temperature jump at the surfaces. The dimensionless value of the jump is $(1/2)/(1 + \delta)$. The hotter surface is at a higher temperature and the cooler surface at a lower temperature than the temperatures of the material near the surfaces.

Note that the solution (19) is quite deficient in details. The temperature of the first wave is higher and of the second wave lower than the temperatures of the surfaces at $\chi = 0$ and $\chi = \delta$, respectively. In the limit

of $\delta \to 0$ we find $\theta_1 \to -\infty$, $\theta_2 \to +\infty$ and $Q \to \infty$, whereas according to the new solution, equation (21), we have in this extreme case $\theta_1 \to 1$, $\theta_2 \to 0$ and $Q \to 1/2$. The fluxes given by equations (19) and (21), being written in dimensional form, show that in the limit of $\tau \to \infty$ and $k/\tau = cv^2$ is non-zero constant, resulting in a pure wave equation, the conventional boundary conditions (18) do not yield a finite steady heat flow for any distance L between the plates, whereas the new boundary conditions (20) yield a finite heat flow for any L.

The heat flux predicted by Fourier's model and by the hyperbolic model with the boundary conditions of the Fourier model, equation (18), also goes to infinity if the distance between the plates L decreases or $\delta \to 0$. The heat flux found with the nonequilibrium boundary conditions (20) in this case is in dimensional form:

$$q_x = \frac{k(T_0 - T_L)}{2(\alpha\tau)^{1/2}} = \frac{cv(T_0 - T_L)}{2}. \quad (22)$$

This expression can be compared to Knudsen's formula for the heat flux in the free molecular regime through a gas contained between two parallel plates [9, 17]. To apply this equation to gases we should know the value of the relaxation time or the velocity of the thermal waves. If we take v as the mean molecular speed in the x-direction, then for an isotropic velocity distribution $v = \bar{v}/2$, where $\bar{v} = (8RT/\pi M)^{1/2}$ is the mean molecular speed at temperature T. For small temperature differences T can be replaced by $(T_0 + T_L)/2$. After this equation (22) becomes:

$$q_x = \frac{Pc_V}{(2\pi MRT)^{1/2}}(T_0 - T_L)$$

where P is the pressure of a gas at the same density as the gas between the plates at a temperature $T = (T_0 + T_L)/2$, R the molar gas constant, M the molecular mass and c_v the molar heat capacity at constant volume.

The energy of gas molecules depends on their speed and therefore the speed $v = \bar{v}/2$ is not representative for the calculation of heat flux. If we take into account the translational energy carried by gas molecules being $4/3$ times as great as the mean translational energy, due to the fact that faster molecules carry more energy [9], we obtain exactly Knudsen's formula [9, 17]:

$$q_x = \frac{P(c_V + R/2)}{(2\pi MRT)^{1/2}}(T_0 - T_L).$$

For arbitrary values of δ the heat flux can be represented in a form:

$$\frac{q}{q_x} = \left(1 + \frac{1}{2}\frac{L}{(\alpha\tau)^{1/2}}\right)^{-1}.$$

Since $(\alpha\tau)^{1/2}$ is of the order of the mean free path λ this expression is similar to that known for the heat flux through a monatomic gas between parallel plates

in the transition region [17], where the mean free path is of the order of the distance between the plates. The discrepancy in coefficients is not surprising in view of the many approximations that were made and the uncertainties in the definition of the mean free path in this ultra-simplified treatment. It is certainly interesting, that already the simplest hyperbolic model describes the heat flux of arbitrary values of the Knudsen number rather well.

Unsteady heat conduction in a slab

Consider a slab initially at a uniform temperature T_{in}. At time $t = 0$ both external surfaces at $x = 0$ and $x = L$ are suddenly exposed to a temperature T_s and kept at that temperature. The analytical solution of the hyperbolic equations suitable for small values of the time can be found by means of a Laplace transform. The dimensionless quantities are defined as follows:

$$\theta = \frac{T - T_{in}}{T_s - T_{in}} \quad Q = \frac{q\sqrt{\alpha\tau}}{k(T_s - T_{in})}$$

$$\chi = \frac{x}{2\sqrt{\alpha\tau}} \quad \varsigma = \frac{t}{2\tau} \quad \delta = \frac{L}{2\sqrt{\alpha\tau}}. \qquad (23)$$

In dimensionless terms the initial values and the conventional and nonequilibrium boundary conditions, respectively, are:

$$\varsigma = 0 \quad \theta = 0 \quad Q = 0 \quad \text{or } \theta_1 = \theta_2 \qquad (24)$$

$$\chi = 0 \quad \theta = 1 \quad \chi = \delta \quad \theta = 1 \qquad (25)$$

$$\chi = 0 \quad \theta_1 = 1 \quad \text{or } \theta + Q = 1$$

$$\chi = \delta \quad \theta_2 = 1 \quad \text{or } \theta - Q = 1. \qquad (26)$$

For the nonequilibrium boundary conditions (26) the temperature of the first wave can be presented in the form:

$$\theta_1(\varsigma, \chi) = \sum_{n=0}^{\infty} (-1)^n [F_n(\varsigma, n\delta + \chi)$$

$$+ F_{n+1}(\varsigma, (n+1)\delta - \chi)]$$

where

$$F_n(\varsigma, z) = H(\varsigma - z) \left\{ \Phi_n(\varsigma, z) \right.$$

$$\left. + \int_z^\varsigma e^{y-\varsigma} [I_0(\varsigma - y) + I_1(\varsigma - y)] \Phi_n(y, z) \, dy \right\}$$

$$\Phi_n(\varsigma, z) = e^{-\varsigma} \left(\frac{\varsigma - z}{\varsigma + z} \right)^{n/2} I_n(\sqrt{\varsigma^2 - z^2})$$

and $H(\zeta - z)$ is the unit step function; I_n is the modified Bessel function of the nth order. The corresponding solution for the equilibrium boundary condition (25) is:

$$\theta_1(\varsigma, \chi) = \sum_{n=0}^{\infty} (-1)^n [F(\varsigma, n\delta + \chi) + \Phi(\varsigma, n\delta + \chi)$$

$$+ F(\varsigma, (n+1)\delta - \chi) - \Phi(\varsigma, (n+1)\delta - \chi)]$$

where $F(\varsigma, z) = F_0(\varsigma, z)$ and $\Phi(\varsigma, z) = \Phi_0(\varsigma, z)$. Due to the symmetry of the problem relative to the centre plane of the slab also $\theta_2(\varsigma, \chi) = \theta_1(\varsigma, \delta - \chi)$ holds.

The temperature distributions calculated with different boundary conditions are shown in Fig. 1 for the case $\delta = 1$, as considered by Taitel [11]. Figure 1(a) shows, in sequential moments of time, the temperatures of the wave moving in the positive x-direction as a function of the dimensionless position. The movement of heat waves can be retraced taking into account the dimensionless wave speed is 1. The behavior of the wave temperature can be easily understood physically when the new, nonequilibrium boundary conditions (26) are used. In this case the temperature of the wave moving to the right at the left surface is equal to the surface temperature everywhere and changes inside the slab due to the energy exchange with the second wave moving in opposite direction. When the conventional boundary conditions (25) are used the temperature of the wave at the surface changes. Its dimensional value is higher than the surface temperature $\theta_s = 1$ during the time intervals $2n < \zeta < 2n + 1$ $(n = 0, 1, \ldots)$ and lower otherwise. At short moments of time, or $\varsigma \to 0$, it is twice as high as the surface temperature. The change of the wave temperature near the surface from the value > 1 to the value < 1 and vice versa occurs when the second wave comes to the surface. Such wave temperature behavior neat the surface can be explained physically only via a miracle that regulates the temperature of the reflected wave so that the average temperature of two waves is equal to the surface temperature. The temperature $\theta = (\theta_1 + \theta_2)/2$ calculated with different boundary conditions is given in Fig. 1(b). As expected there are no physical contradictions if the nonequilibrium boundary conditions (26) are used, whereas with the old boundary conditions (25) a superposition of two waves results in a temperature exceeding the initial, $\theta_{in} = 0$, and boundary, $\theta_s = 1$, temperatures, as can be seen at times $\varsigma = 0.7$ and $\varsigma = 1.4$.

In recent papers, Bai and Lavine [10, 12] considering the same problem have found that for the case of slab cooling or $T_s < T_{in}$ the temperature in the layer may be lower than the absolute zero. The use of the new boundary conditions (26) avoids this physically unrealistic prediction. The temperature in the centre of the layer, as time elapses, calculated with different boundary conditions for the case considered by Bai and Lavine [12] is presented in Fig. 2. An important feature of the new solution is that the temperature in any point is a monotonic function of time [see Fig. 2(b)], whereas the old solution gives temperature oscillations of decreasing amplitude as shown in Fig. 2(a). These results can be understood

(a)   **(b)**
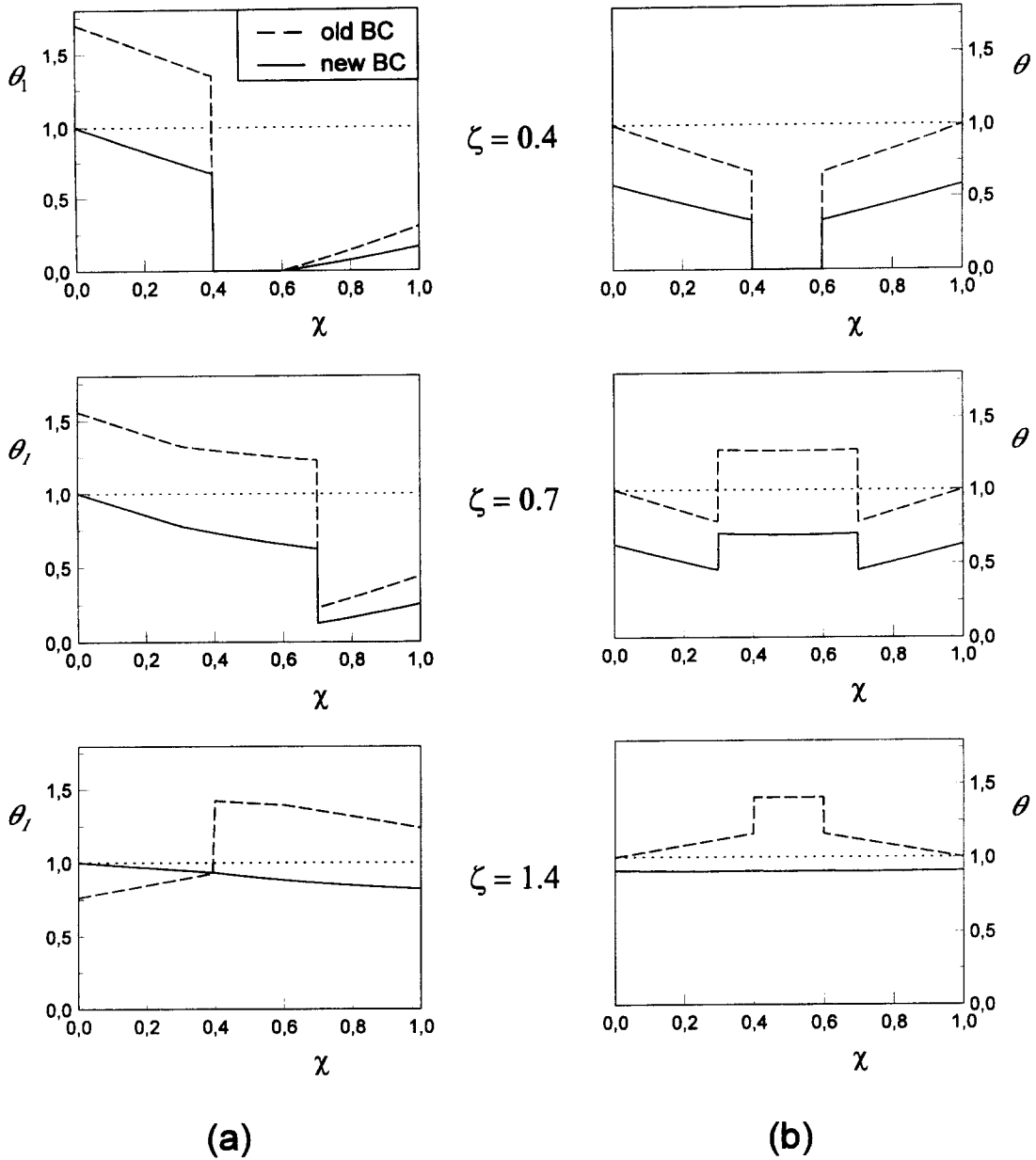
Fig. 1. Thermal wave propagation in a slab after a sudden temperature change at its surfaces as predicted by the hyperbolic model with equilibrium and nonequilibrium boundary conditions: (a) temperature of the wave moving to the right; (b) temperature in the slab.

through consideration of the wave temperatures predicted with different boundary conditions as discussed in the previous example.

The use of the new boundary conditions also result in an essentially different behavior and in different values of the heat fluxes across the surfaces for short times, that is for times where the application of the hyperbolic model instead of Fourier's law may be especially important, as is shown in Fig. 3 for the case considered by Bai and Lavine [12]. The heat flux found with the conventional boundary conditions (25) periodically changes in sign and with the new boundary conditions (26) it monotonically decreases. Figures 4 and 5 compare the heat flux at the surface of

the semi-infinite slab ($\delta \to \infty$) and the total quantity of heat $U(\varsigma) = \int_0^\varsigma Q(0, \varsigma) \, d\varsigma$, which till time ζ has crossed this surface as a function of time as predicted by the different models. At short times, the fluxes found with the old and new boundary conditions are, respectively:

$$Q(0, \varsigma) = 1 - \varsigma + \tfrac{3}{4}\varsigma^2 + O(\varsigma^3)$$

$$Q(0, \varsigma) = \frac{1}{2} - \frac{\varsigma}{4} + \frac{\varsigma^2}{8} + O(\varsigma^3)$$

and differ a factor two for $\zeta \to 0$. For $\zeta \to \infty$ all solutions reduce to the Fourier solution. The total amount
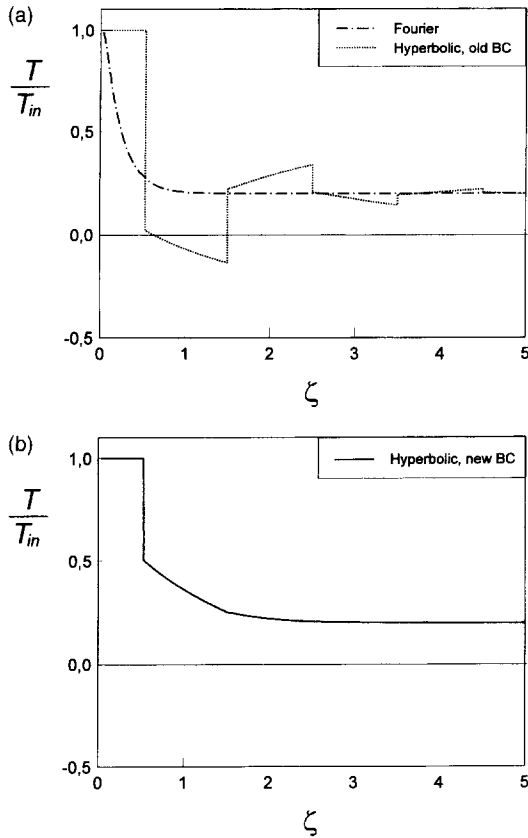
(a)

(b)

Fig. 2. Temperature at the centre of the slab for $\delta = 1$ and $T_s/T_{in} = 0.2$: (a) Fourier and hyperbolic models with equilibrium boundary conditions; (b) hyperbolic model with nonequilibrium boundary conditions.
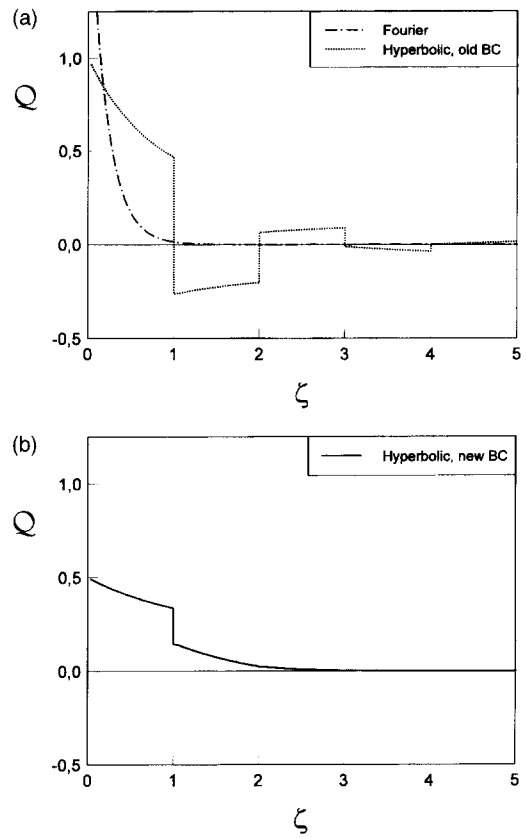
(a)

(b)

Fig. 3. Heat flux at a slab surface for $\delta = 1$ and $T_s/T_{in} = 0.2$: (a) Fourier and hyperbolic models with equilibrium boundary conditions; (b) hyperbolic model with nonequilibrium boundary conditions.

of heat which has crossed the surface $x = 0$ according to the Fourier model differs less than 1% from that of the hyperbolic model with the new boundary conditions after a time $t \approx 8 \cdot 10^3 \tau$. If the equilibrium boundary conditions are used for the hyperbolic equations the difference between the hyperbolic and Fourier models vanishes in this sense already at $t \approx 26\tau$.

It is of great interest to test the models using experimental data. Unfortunately, almost no work has been reported to validate directly the hyperbolic nature of heat conduction by comparing experimentally observed temperature or flux distributions with non-Fourier predictions. A recent paper of Mitra *et al.* [8] undoubtedly is of much interest. The authors have found a perfect agreement between experiments on heat propagation in processed meat and the predictions with the equations (1) and (4), when two or three identical meat samples at different initial temperatures are brought into contact with each other. They also presented experimental results on heat propagation in a cold meat sample brought into contact with a warm aluminum plate of a constant temperature. In this experiment the temperature measured at a distance 6.6 mm from the plate significantly differs from the theoretical hyperbolic heat conduction curve calculated by the authors [Fig. 6(a)], whereas the mea-
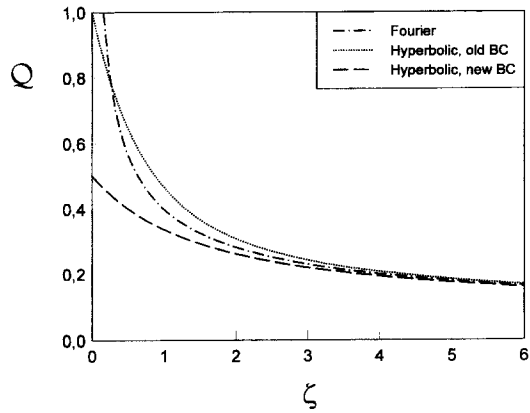

Fig. 4. Heat flux at the surface of a semi-infinite slab as a function of time, predicted by different models.

sured temperatures match the theoretical prediction very well at a distance at 14.0 mm from the plate [Fig. 6(b)]. This experiment with the constant temperature plate is of much interest in the context of the present paper. The dimensionless temperatures at the two positions calculated with the new boundary conditions:

$$\chi = 0 \quad \theta_1 = 1 \quad \text{or } \theta + Q = 1$$

$$\chi \to \infty \quad \theta_2 = 0 \quad \text{or } \theta - Q = 0$$
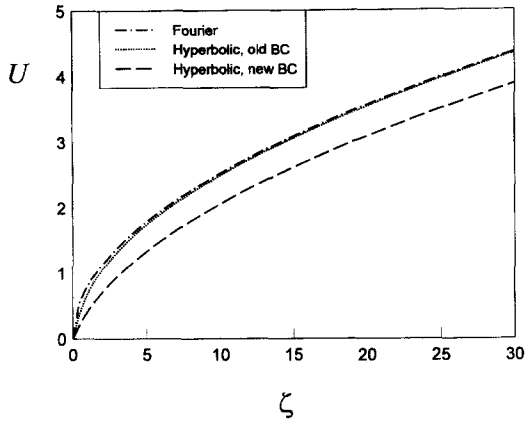
Fig. 5. Total amount of heat which has crossed the surface of a semi-infinite slab as a function of time, predicted by different models.
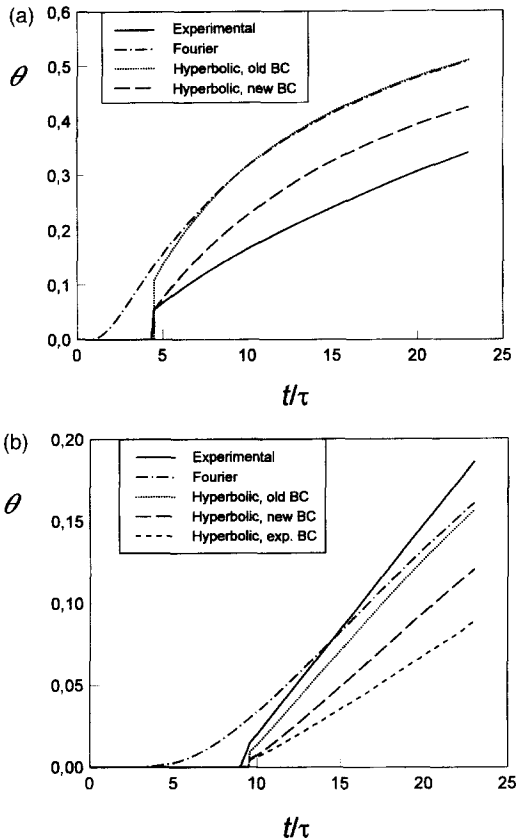

Fig. 6. Temperature response at (a) $\chi = 4.48$ ($x = 6.6$ mm) and (b) $\chi = 9.50$ ($x = 14.0$ mm) in a meat sample brought into contact with a constant temperature plate as calculated by different models, compared to the experimental data of Mitra et al. (1995). The lower curve in (b) is the solution of equation (5) with the experimental curve from (a) used as a boundary condition.

are given in Fig. 6 compared to the experimental results, to the predictions of the Fourier model and of the hyperbolic model with the old boundary conditions. The use of the new boundary condition improves the theoretical curve for the first point [Fig.

6(a)] and makes it worse for the second point [Fig. 6(b)].

Unfortunately, the experimental temperatures of Mitra et al. [8], reproduced in Fig. 6, cannot be reconciled with equations (1) and (4). If these equations hold, we are able to calculate the temperature at the second point through the temperature at the first point, because the temperature of the meat sample at $x > 6.6$ mm is equal to the initial temperature when the heat front reaches position $x = 6.6$ mm. Calculated in such a way the temperature at $x = 14.0$ mm, the lower curve in Fig. 6(b), essentially differs from the experimental curve [see Fig. 6(b)]. This is rather strange in view of the results obtained by the authors with identical meat samples, brought into firm contact.

Some confirmation of the nonequilibrium boundary conditions follows from a different observation of Mitra et al. [8] in the same experiment. They observed the inability of the cooling mechanism to instantaneously attain a constant interface temperature between the meat sample and the aluminum plate. During the initial phase of the experiment it was found to be $4-5°C$ lower than the set constant plate temperature. According to the new solution the dimensionless temperature of the meat sample near the plate varies as:

$$\theta(0, \varsigma) = 1 - \tfrac{1}{2}e^{-\varsigma}[I_0(\varsigma) + I_1(\varsigma)].$$

At short times we have:

$$\theta(0, \varsigma) = \frac{1}{2} + \frac{\varsigma}{4} - \frac{\varsigma^2}{8} + O(\varsigma^3). \qquad (27)$$

In the experiment the initial temperature $T_{in} = 8.1°C$ and the temperature of the plate $T_s = 28.2°C$. According to equations (23) and (27) $T(0, t) - T_{in} \approx (T_s - T_{in})/2$ at short times, that is the temperature of the meat sample near the plate is $10°C$ lower than the plate temperature. It is impossible to say what temperature was measured by the thermocouple situated between the meat sample and the aluminum plate, but the measured temperature of $4-5°C$ less than the set constant plate temperature corresponds to the average temperature of the aluminum plate and the meat sample predicted by equation (27). To clarify the governing heat transport mechanisms and the boundary conditions in the processed meat, further experiments on this point seem to be worthwhile. Steady-state measurements of thermal heat conductivity with meat samples of varying thickness can also be very useful.

DISCUSSION AND CONCLUSIONS

The instantaneous local thermal equalization and the infinite speed of heat propagation implied in the classical theory of heat conduction hamper a realistic physical picture of heat conduction phenomena. A clearer picture of heat conduction is obtained if the local nonequilibrium state is represented explicitly.

This approach provides a simple interpretation of the well known hyperbolic equations (1) and (4). These equations have remarkable features. Both equations have the same appearance: they are energy equations for a local nonequilibrium system, and the heat flux in the hyperbolic equations is a second state variable, allowing us to distinguish between states with the same global temperature but different local energy distributions. The dimensionless value of the heat flux $q/(cTv)$ seems to be a natural measure of the departure from a local thermal equilibrium which can be regarded as the accompanying equilibrium state introduced by Kestin [13].

The appearance of the heat flux as a second state variable agrees with the generalized theory of extended irreversible thermodynamics, which introduces dissipative fluxes as new independent variables [14]. Moreover, the microscopic local nonequilibrium consideration indicates that the hyperbolic heat conduction equations require not only an additional initial condition, compared to the Fourier model, but also different boundary conditions at the interface between two contacting bodies. Physically realistic nonequilibrium boundary conditions follow immediately from the same basis as the governing equations. Unfortunately, the available experimental data are not sufficiently accurate to conclude what boundary conditions, equilibrium or nonequilibrium, are appropriate for the hyperbolic model. Nevertheless, the solutions of equations (1) and (4) with nonequilibrium boundary conditions do not produce the unrealistic results as are obtained with the boundary conditions taken from the classic Fourier model. It should be noted that the hyperbolic model does not preclude transient temperatures higher than the initial and boundary temperatures if the initial heat flux is not zero, but negative absolute temperatures or temperatures as presented in Fig. 1 for equilibrium boundary conditions are physically not allowed: this can be understood from the heat conduction equations in the form of equations (12) and (13) and their physically appropriate boundary conditions. This is in contrast to opinions met within the literature. For example, according the Özisik and Tzou [4] the temperature behavior reported by Taitel [11] and presented in Fig. 1 is admissible within the framework of the thermal wave model. Barletta and Zanchini [18] also did not discover a physical contradiction in these results arguing that the calculated temperature does not represent the thermodynamic temperature.

The heat conduction model of equations (1) and (4) with the nonequilibrium boundary conditions of equations (15) or (16) combines the ideas of a state of local nonequilibrium embedded in the known hyperbolic equations and the jump boundary conditions. This model gives a new insight in the temperature jump effect in gases. According to the existing theory, the real temperature is continuous and the effect is an apparent discontinuity in the temperature at a gas–solid interface when heat is flowing across the bound-

ary due to a more rapid temperature variation at a location within several molecule free paths of the surface [19]. In the hyperbolic model the temperature jump effect is an inherent property and a real effect, arising because only the reflected molecules can be influenced directly by the wall. Moreover, the temperature jump boundary condition (6) is derived only when the molecular mean free path is small compared to the characteristic length [9, 17]. Equations (12) and (13) and their boundary conditions, equations (15) or (16) do not require such limitation.

Numerous calculations with the conventional boundary condition (3) [20–22], as well as the results given in this paper, show that non-Fourier effects are important only for transient thermal processes with characteristic times of the order of magnitude of the relaxation time. Hyperbolic equations with the new, nonequilibrium boundary conditions give results significantly different from those obtained with the old boundary conditions. Non-Fourier effects may be important for time scales much larger than the relaxation time. These effects appear also in steady-states when the characteristic time, the time of interaction between waves, is comparable or smaller than the relaxation time.

Finally, it is worth noting that hyperbolic heat conduction models should not only be associated with heat transfer problems in solids, liquids and gases at exotic situations involving very short times, extreme thermal gradients or temperatures near absolute zero. For many practical problems concerning heat transfer in multiphase systems and for complicated hydrodynamic conditions, Fourier's law with a kind of effective heat conductivity is commonly accepted, assuming heat transport is diffusionlike [23]. In these cases the carriers of thermal energy, being usually the fluid elements are characterized by velocities, mean free paths and mean free paths times, which are incomparable with those of uniform solids, liquids and gases at normal conditions. In such situations the applicability of the Fourier law becomes rather questionable. One of the pronounced examples is Taylor or shear heat dispersion [24, 25]. In the classic case of axial dispersion in laminar flow in a tube of a diameter d_t the heat relaxation time is $\tau = d_t^2/(60\alpha)$ [26] and thermal equilibrium is established only at distances much larger than $(RePr/60)d_t$, where Re and Pr are the Reynolds and Prandtl numbers. Another important practical example is the axial and radial heat dispersion in a fluid flowing through a tube or rectangular channel filled with solid particles [23, 27]. The application of the hyperbolic equations to such systems is intriguing in view of the wave character of heat and mass propagation at high Reynolds numbers [28]. The last example is attractive as an object for the experimental verification of the boundary conditions discussed in the present paper, because of the long relaxation time. The temperature-jump phenomenon is clearly seen when the ratio of the tube diameter or

the distance between the plates to the particle diameter is not too large, say 6–10 [29].

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